Problems PreCal 1508 PLTL Workshop, October 25, 2011
PLs: Alex Knaust, Edith Mejia. Lecturer: Yi-Yu Liao

These problems represent a review of the content that will be covered by the third exam. The answers are available as a pdf at `http://alex.knaust.info/pltlfall2011/`

1a) We will start with the left side because it has more terms, and it is generally simpler to condense an expression than expand it. Each step should be very simple and easy to follow, and you should note which identities are used.

$$
\begin{array}{rcl}
& \cos x(\tan^2 x + 1) & \text{Problem statement} \\
= & \cos x(\sec^2 x) & \text{Used the identity } tan^2 x + 1 = \sec^2 x \\
= & \cos x \frac{1}{\cos^2 x} & \text{sec reciprocal of cos} \\
= & \frac{1}{\cos} & \\
= & \sec x & \text{Desired result}
\end{array}
$$

1c)

$$
\begin{array}{rcl}
& \tan\left(\frac{\pi}{4} - \theta\right) & \text{Problem statement} \\
= & \frac{\tan\left(\frac{\pi}{4}\right) - \tan\theta}{1 + \tan\left(\frac{\pi}{4}\right)\tan\theta} & \text{tangent difference identity} \\
= & \frac{1 - \tan\theta}{1 + \tan\theta} & \tan\left(\frac{\pi}{4}\right) = 1
\end{array}
$$

1e)

$$
\begin{array}{rcl}
& \frac{\cos 3\beta}{\cos\beta} & \text{Problem statement} \\
= & \frac{\cos\beta + 2\beta}{\cos\beta} & \\
= & \frac{\cos\beta\cos 2\beta - \sin\beta\sin 2\beta}{\cos\beta} & \text{cosine sum identity} \\
= & \frac{\cos\beta\cos 2\beta}{\cos\beta} - \frac{\sin\beta\sin 2\beta}{\cos\beta} & \text{Split up the numerator} \\
= & \cos 2\beta - \frac{\sin\beta(2\sin\beta\cos\beta)}{\cos\beta} & \text{Sine double angle identity} \\
= & 1 - 2\sin^2\beta - 2\sin^2\beta & \text{Cosine double angle identity } \cos 2u = 1 - 2\sin^2 u \\
= & 1 - 4\sin^2\beta & \text{Desired result}
\end{array}
$$

1f) This one is a bit tricky because it takes a creative multiplication by one to go straight from the initial problem to the result. However sometimes it is necessary to do some trial and error and hard thinking to arrive at a step, *this is perfectly normal*!

I thought of this step just by thinking about how I could turn the $\tan x \tan y$ in the numerator of the left-hand-side into the -1 of the right hand side, and tried to see if that would solve the other problems as well

$$\begin{aligned}
& \tfrac{\tan x+\tan y}{1-\tan x\tan y} && \text{Problem statement.} \\
= & \tfrac{\tan x+\tan y}{1-\tan x\tan y}\cdot\tfrac{\frac{1}{\tan x\tan y}}{\frac{1}{\tan x\tan y}} && \text{Multiplication by 1} \\
= & \tfrac{\frac{\tan x+\tan y}{\tan x\tan y}}{\frac{1-\tan x\tan y}{\tan x\tan y}} && \\
= & \tfrac{\frac{\tan x}{\tan x\tan y}+\frac{\tan y}{\tan x\tan y}}{\frac{1}{\tan x\tan y}-\frac{\tan x\tan y}{\tan x\tan y}} && \text{break up the numerators} \\
= & \tfrac{\frac{1}{tan y}+\frac{1}{\tan x}}{\frac{1}{\tan x}\cdot\frac{1}{\tan y}-1} && \text{cancel some terms} \\
= & \tfrac{\cot y+\cot x}{\cot x\cot y-1} && \text{Use } \cot u=\tfrac{1}{\tan u}
\end{aligned}$$

2a) Since we are solving for x it seems that it would be a lot more helpful if we only had one function to work with, not two. This can be accomplished with a simple identity, and you can work by either converting sin^2x to $1-cos^2x$ or the other way around.

$$\sin^2 x = 3\cos^2 x \Longleftrightarrow 1-\cos^2 x = 3\cos^2 x$$

Now we want to isolate the unknowns on one side,

$$\Longleftrightarrow 1-4cos^2x = 0 \Longleftrightarrow \cos^2 x = \frac{1}{4}$$

Taking the square root (considering that the answer could be positive or negative we have

$$\Longleftrightarrow \cos x = \pm\frac{1}{2}$$

So we ask ourselves where this occurs in the interval $[0, 2\pi)$ on the unit circle, or alternatively we take the $arccos(\frac{1}{2})$ and translate it into the other coordinates. From the unit circle it is known that

$$\cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Since the question asks for all solutions, we have to find a form for all coterminal angles (since the same thing will occur every period, for example $\cos\left(\frac{2\pi}{3}+2\pi\right)$) Thus a general form for our solutions might be

$$x = \frac{2\pi}{3}+n\pi, \quad \frac{\pi}{3}+n\pi$$

Where n denotes any integer $0, 1, 2, 3, ...$

2e) As before we first transform this so that we only have one type of function A useful identity tells us that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

$$\sin\left(x + \frac{\pi}{2}\right) - cos^2 x = 0 \iff \cos x - \cos^2 x = 0$$

Since it is in a form equal to 0, we know that if the left hand side was a product would could break it up because each of the factors might be equal to 0, So we try to factor the left hand side

$$\iff \cos x(1 - \cos x) = 0 \implies \cos x = 0 \text{ or } \cos x = 1$$

Which gives the solutions

$$x = 2n\pi, \quad \frac{\pi}{2} + n\pi$$

3b) Here we can simply evalute the functions in order

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

Since arccos is the (restricted) inverse of cos we ask ourselves which angle has $\cos x = \frac{\sqrt{3}}{2}$ with the additional consideration that the range and domain of arccos is restricted

3c) With the arcsin in the inside, the best step is to draw a triangle. So we can have a right triangle with hypotenouse 5 and side opposite of the angle returned by $\arcsin\left(\frac{3}{5}\right)$ is 3 (since sin is opposite over hypotenouse. The other side is $\sqrt{5^2 - 3^2} = 4$

So the tangent of this angle is the opposite side which has length 3 over the adjacent side which has length 4. So

$$\tan\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{3}{4}$$

4a) Just remember the standard form of a (trigonometric) equation $f(x) = a\cos bx - c + d$. For our function this implies that $a = -2, b = 1, c = -\frac{\pi}{4}, d = 0$. (Note that since the standard form has $bx - c$ our c is actually negative).

Then the amplitude (how much the graph stretches vertically) is $|a| = 2$ The phase shift is given by $\frac{c}{b} = -\frac{\pi}{4}$ The period is $\frac{2\pi}{b} = 2\pi$

If we want to look at the standard cycle of cos (from 0 to 2π) and how our function has transformed this, we figure out where it has moved to, so the left and right endpoints of a cycle can be found by solving $bx - c = 0$ and $bx - c = 2\pi$. So we have the interval $[-\frac{\pi}{4}, \frac{11\pi}{4}]$

Since we are multiplying with $a = -2$ we flip the cos in this interval, so the maximum, originally at $(0, 1)$ is now a minimum at $(-\frac{\pi}{4}, -2)$ and the minimum which was at $(\pi, -1)$ is now a maximum at $(\pi - \frac{\pi}{4}, 2)$.

These should be enough points to draw a moderately accurate graph. (use WolframAlpha or your calculator)

5a) Word problems are best solved by first drawing a triangle or diagram to represent the information given, unfortunately that is not possible in this context, but you should do so to solve such a problem. We don't know that the triangle is a right triangle (and checking pythagorea's theorem shows that it isn't) So we ask ourselves which angle should be the largest. Well the largest angle has to opposite of the largest side, it wouldn't make much sense otherwise. Thus the angle we are looking for is between the sides of length $5m$ and $7m$.

Since we know all three sides and only one angle, the *Law of Cosines* seems to best fit our needs. We use the form that will make use of the angle we are looking for so

$$9^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cdot \cos L$$

(L denotes the largest angle)

$$\iff \frac{9^2 - 5^2 - 7^2}{-2 \cdot 5 \cdot 7} = \cos L$$

We can find L simply by using arccos to extract it.

$$\iff \arccos\left(-\frac{7}{70}\right) = L$$

$$L \approx 95.74^\circ$$

Which makes sense, because the other angles have to be smaller than it so that they sum up to 180° and we know that it is the largest angle.