Problems PreCal 1508 PLTL Workshop, October 25, 2011
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These problems represent are select solutions to the problems from the review sheet for the second exam (*review2_10-25-2011pltl.tex*). The questions are available as a pdf at `http://alex.knaust.info/pltlfall2011/`

Contact me at awknaust@miners.utep.edu If there are any inconsistencies in the solutions or if you have any questions about the content of the second Exam in general.

1a) We can immediately see what the horizontal asymptote is by examining the leading coefficient and the degree of the numerator ($x^2 - 5x + 4$) and denominator ($x^2 - 16$). Since the degree of the denominator is 2, and the degree of the numerator is the same, we know there is a horizontal asymptote and it is at $y = \frac{1}{1} = 1$ since the leading coefficients of the numerator and denominator are both 1.

The next step is to factor the denominator and numerator to gain insight about the domain, vertical asymptotes and holes.

$$f(x) = \frac{x^2 - 5x + 4}{x^2 - 16} = \frac{(x-4)(x-1)}{(x-4)(x+4)}$$

We now know the following

- The factor $(x - 4)$ in both the numerator and denominator indicates there is a hole at $x = 4$
- The factor $(x + 4)$ only in the denominator indicates there is a vertical asymptote at $x = -4$
- The factor $(x - 1)$ in the numerator only indicates that f has an x-intercept at $(1, 0)$

Determining the y intercept is as simple as finding $f(0) = \frac{0^2-5\cdot0+4}{0^2-16} = -\frac{4}{16} = -\frac{1}{4}$.

The domain of a rational function is all real numbers except the zeros of the denominator. Since the denominator has zeros at $x = 4, x = -4$ the domain of f is $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$ or $\mathbb{R} \setminus \{-4, 4\}$

You can view a rendering of the graph at WolframAlpha

1c) The degree of the numerator is less than that of the denominator, so h has a horizontal asymptote at $y = 0$.

The denominator has no real zeros, and thus cannot be factored in the real numbers. This means there are neither vertical nor slant asymptotes, nor holes, and that the domain is $\mathbb{R}$.

h has a y-intercept at $h(0) = \frac{4\cdot0}{0^2+4} = 0 \Longrightarrow (0, 0)$. Obviously this is also the $x - intercept$

You can view a rendering of the graph at WolframAlpha

2a) Familiarize yourself with the graph of the standard logarithmic function which has an x-intercept at $(1, 0)$ and a vertical asymptote at $x = 0$. For $f(x) = \log_3(x - 4) + 2$ we need to shift the graph 4 units to the right and 2 units up. This means the vertical asymptote is moved to $x = 4$ and the domain is now $x > 4$.

To find the x-intercepts we solve for

$$0 = \log_3(x - 4) + 2 \Longleftrightarrow 3^{-2} = (x - 4) \Longleftrightarrow x = \frac{1}{9} + 5$$

And there is obviously no y-intercept since $x = 0$ is not in the domain.

Yielding the following graph

3a) Product rule and power rule yields : $\log_3 x + 2\log_3 x = \log_3 x + \log_3 x^2 = \log_3 x^3$

3c) Quotient rule yields : $\ln 19 - \ln x = \ln \frac{19}{x}$

3e) Change of base yields : $\frac{\ln(4x)}{\ln 5} = \log_5(4x)$

4a) Use the *one-to-one* property by rewriting 32 as 2^5, or take $\log_2$ of both sides yielding

$$2^{x-3} = 32 \iff \log_2(2^{x-3}) = \log_2 32 \iff x - 3 = 5 \iff x = 8$$

4d) First step is to remove the logarithm by exponentiating both sides with base e

$$\ln\sqrt{x+8} = 3 \iff e^{\ln\sqrt{x+8}} = e^3 \iff \sqrt{x+8} = e^3 \iff x + 8 = (e^3)^2 \iff x = e^6 - 8$$

Which is in the domain of the original equation.

4e) Combine the logarithms into one logarithm on each side, so that we can use the *one-to-one* property

$$\log_6(x+2) - \log_6 x = \log_6(x+5) \iff \log_6 \frac{x+2}{x} = \log_6(x+5) \iff \frac{x+2}{x} = x + 5$$

Now we can transform this into a quadratic equation in x and solve for x

$$\frac{x+2}{x} = x + 5 \iff x + 2 = x^2 + 5x \iff 0 = x^2 + 4x + -2$$

Apply the quadratic formula

$$\iff x = \frac{-4 \pm \sqrt{24}}{2} \iff x = -2 \pm \sqrt{6}$$

Lets check if they are in the domain of the original equation... $-2+\sqrt{6} > 0$ so that should be OK. $-2-\sqrt{6} < 4$ So it won't work with $\log_6(x+2)$ in the Left Hand Side of the original equation, so $x = -2 + \sqrt{6}$ is the only solution

5a) It is my strong recommendation that you double check your arithmetic whilst doing any sort of systems of equations solving, and writing down the exact row operations you performed can aid you in doing this.

To solve with Gauß-Jordan the procedure is as follows

(a) Write the system of equations as an augmented matrix

(b) Bring the matrix into row echelon form using row operations

(c) Bring the matrix into reduced row echelon form using row operations (Coefficient matrix will be I_n)

The system of equations gives us the following matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{array}\right]$$

We want to eliminate the coefficient of x in R_3 (7) first. To do this we make use of R_1 as follows

$$R_3 = -7R_1 + 2R_3 \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 0 & -3 & -21 & -156 \end{array}\right]$$

Next remove the y coefficient in R_3 (2).

$$R_3 = 3R_2 + 2R_3 \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 0 & 0 & -45 & -270 \end{array}\right]$$

We now have 0's below the diagonal which was the first step of our plan. Now simplifying R_3 gives us

$$R_3 = R_2 - R_3 \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

We want to achieve 0's everywhere except the diagonal so we continue using R_3 to work on the other rows.

$$R_2 = \frac{R_3 + R_2}{2} \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 24 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{array}\right] \qquad R_1 = R_1 - 3R_3 \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 6 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$R_1 = \frac{R_1 + R_2}{2} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Which completes the Gauß-Jordan elimination, giving us the results $x = 8$ and $y = 10$ and $z = 6$.

7a) To decompose this into partial fractions, we first check the degree of the numerator and the degree of the denominator. Since the degree of the numerator (0) is less than the degree of the denominator (2), the expression is proper, and we can proceed with the partial fraction decomposition

Once we know the expression is a proper fraction the first step is to factor the denominator,

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)}$$

Since there are two linear factors (of degree 1) in the denominator, we know it can be split up as follows for some constants $A, B \in \mathbb{R}$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Multiplying both sides by the denominator of the left hand side simplifies the equation a bit, giving us

$$1 = A(x+1) + B(x)$$

Now we have two options for solving this

(a) Since the two must be equal as functions (i.e. the equation must hold for all x, we can test some specific values of x that simplify the work of finding A and B

(b) Since the two are also polynomials, in order to be equal they must have the same coefficients, so we can derive a system of equations by grouping the terms on the Right hand side and comparing them with those on the left

For this problem we will use the first method. It is obvious that $x = 0$ and $x = -1$ are useful to us as they remove one of the variables from the equation, so we can use them to find A and B very quickly

$$x = 0 \Longrightarrow 1 = A(0+1) + B(0) \Longleftrightarrow 1 = A$$
$$x = -1 \Longrightarrow 1 = A(-1+1) + B(-1) \Longleftrightarrow B = -1$$

So the answer is

$$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$$

We can check this by combining the Right hand side...

7c) The expression is proper and the denominator seems to be factored. However we want to make sure that x^2-2x+3 is an irreducible polynomial, so we test its discriminant b^2-4ac. $(-2)^2-4(1)(3) = -8$. The discriminant is negative, affirming that it is indeed irreducible. Okay, then we continue with the decomposition considering there is one **linear** and one **quadratic** factor

$$\frac{x^2-4x+7}{(x+1)(x^2-2x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+3}$$

$$x^2-4x+7 = A(x^2-2x+3) + (Bx+C)(x+1)$$

Now we will try comparing the coefficients to find the values of A, B and C. The first step is to expand the Right Hand Side

$$x^2 - 4x + 7 = Ax^2 - 2Ax + 3A + Bx^2 + Bx + Cx + C = Ax^2 + Bx^2 + Bx + Cx - 2Ax + C + 3A$$

$$\iff (1)x^2 - 4x + 7 = (A+B)x^2 + (B + C - 2A)x + (C + 3A)$$

Equating the coefficients gives us the following system of equations

$$\begin{cases} A + B = 1 \\ B + C - 2A = -4 \\ C + 3A = 7 \end{cases}$$

Solving gives $A = 2$, $B = -1$, $C = 1$

So the answer is

$$\frac{x^2 - 4x + 7}{(x+1)(x^2 - 2x + 3)} = \frac{2}{x+1} + \frac{-x+1}{x^2 - 2x + 3}$$

8a)

$$3B - 2D = 3 \cdot \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} - 2 \cdot \begin{bmatrix} 3 & 9 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 & 3 \cdot 1 \\ 3 \cdot 1 & 3 \cdot 4 \end{bmatrix} - \begin{bmatrix} 2 \cdot 3 & 2 \cdot 9 \\ 2 \cdot (-5) & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 & -15 \\ 13 & 4 \end{bmatrix}$$

8b)

$$A \cdot C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 6 \\ -4 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 9 & 0 \end{bmatrix} = \cdot \begin{bmatrix} 1 \cdot 1 - 1 \cdot 0 + 9 \cdot 2 & 2 \cdot 1 + 3 \cdot 0 + 0 \cdot 2 \\ 1 \cdot 2 - 1 \cdot 1 + 9 \cdot 6 & 2 \cdot 2 + 3 \cdot 1 + 0 \cdot 6 \\ 1 \cdot (-4) - 1 \cdot 2 + 9 \cdot (-3) & 2 \cdot (-4) + 3 \cdot 2 + 0 \cdot (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 2 \\ 55 & 7 \\ -33 & -2 \end{bmatrix}$$

8e) The determinant $\det(D)$ is $3 \cdot 4 - 9 \cdot (-5) = 57$. Thus the inverse of D is

$$D^{-1} = \frac{1}{57} \begin{bmatrix} 4 & -9 \\ 4 & 3 \end{bmatrix}$$

8g) The determinant of a 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $ad - bc$. So in this case we have $det(B) = (2)(4) - (1)(1) = 7$. Which tells us the matrix is nonsingular, i.e. that it is invertible.